Histograms and Wavelets on Probabilistic Data

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Abstract— **There is a growing realization that uncertain information is a first-class citizen in modern database management. As such, we need techniques to correctly and efficiently process uncertain data in database systems. In particular, data reduction techniques that can produce concise, accurate synopses of large probabilistic relations are crucial. Similar to their deterministic relation counterparts, such compact probabilistic data synopses can form the foundation for human understanding and interactive data exploration, probabilistic query planning and optimization, and fast approximate query processing in probabilistic database systems.**

In this paper, we introduce definitions and algorithms for building histogram- and Haar wavelet-based synopses on probabilistic data. The core problem is to choose a set of histogram bucket boundaries or wavelet coefficients to optimize the accuracy of the approximate representation of a collection of probabilistic tuples under a given error metric. For a variety of different error metrics, we devise efficient algorithms that construct optimal or near optimal size B histogram and wavelet synopses. This requires careful analysis of the structure of the probability distributions, and novel extensions of known dynamic-programming-based techniques for the deterministic domain. Our experiments show that this approach clearly outperforms simple ideas, such as building summaries for samples drawn from the data distribution, while taking equal or less time.

I. Introduction

Modern real-world applications generate massive amounts of data that is often *uncertain and imprecise*. For instance, data integration and record linkage tools can produce distinct degrees of confidence for output data tuples (based on the quality of the match for the underlying entities) [1]; similarly, pervasive multi-sensor computing applications need to routinely handle noisy sensor/RFID readings [2]. Motivated by these new application requirements, recent research on *probabilistic data management* aims to incorporate uncertainty and probabilistic data as "first-class citizens" in the DBMS.

Among different approaches for modeling uncertain data, *tuple- and attribute-level uncertainty models* have seen wide adoption both in research papers as well as early system prototypes [3], [4], [5]. In such models, the attribute values for a data tuple are specified using a probability distribution over different mutually-exclusive alternatives (that might also include *non-existence*, i.e., the tuple is not present in the data), and assuming independence across tuples. The popularity of tuple-/attribute-level uncertainty models is due to their ability to describe complex dependencies while remaining simple to represent in current relational systems, with intuitive query semantics. In essence, a probabilistic database is a concise representation for a probability distribution over an exponentially-large collection of *possible worlds*, each representing a possible "grounded" (deterministic) instance of the database (e.g., by flipping appropriately-biased independent coins to select an instantiation for each uncertain tuple). This "possible-worlds" semantics also implies clean semantics for queries over a probabilistic database—essentially, the result of a probabilistic query defines a distribution over possible query results across all possible worlds [1]. The goal of query answering over uncertain data is to provide the expected value of the answer, or tail bounds on the distribution of answers.

Unfortunately, despite its intuitive semantics, the paradigm shift towards tuple-level uncertainty also implies computationally-intractable *#P-hard data complexity* even for simple query processing operations [1]. These negative complexity results for query evaluation raise serious practicality concerns for the applicability of probabilistic database systems in realistic settings. One possible avenue of attack is via *approximate query processing* techniques over probabilistic data. As in conventional database systems, such techniques need to rely on effective *data reduction methods* that can effectively compress large amounts of data down to concise *data synopses* while retaining the key statistical traits of the original data collection [6]. It is then feasible to run more expensive algorithms over the much compressed representation, and obtain a fast and accurate answer. In addition to enabling fast approximate query answers, such compact synopses can also provide the foundation for human understanding and interactive data exploration, and probabilistic query planning and optimization.

The data-reduction problem for deterministic databases is well understood, and several different synopsis construction tools exist. *Histograms* [7] and *wavelets* [8] are two of the most popular general-purpose data-reduction methods for conventional data distributions. It is hence meaningful and interesting to build corresponding histogram and wavelet synopses of probabilistic data. Here, histograms, as on traditional "deterministic" data, divide the given input data into "buckets" so that all tuples falling in the same bucket have similar behavior: bucket boundaries are chosen to minimize a given error function that measures the within-bucket dissimilarity. Likewise, wavelets represent the probabilistic data by choosing a small number of wavelet basis functions which best describe the data, and contain as much of the "expected energy" of the data as possible. So, for both histograms and wavelets, the synopses aim to capture and *describe* the probabilistic data as accurately as possible given a fixed size of synopsis. This description is of use both to present to users to compactly show the key components of the data, as well as in approximate query answering and query planning. Clearly, if we can find approximate representations that compactly represent a much larger data set, then we can evaluate queries on these summaries to give an approximate answer much more quickly than finding the exact answer. As a more concrete example, consider the problem of estimating the expected cardinality of a join of two probabilistic relations (with a join condition on two uncertain attributes): this can be a very expensive operation, as all possible alternatives for potentially joining uncertain tuples (e.g., OR-tuples in Trio [4]) need to be carefully considered. On the other hand, the availability of concise, low-error histogram/wavelet synopses for probabilistic data can naturally enable fast, accurate approximate answers to such queries (in ways similar to the deterministic case [8], [9]).

Much recent work has explored many different variants of the basic synopsis problem, and we summarize the main contributions in Section II. Unfortunately, this work is not applicable when moving to *probabilistic* data collections that essentially represent a huge set of possible data distributions (i.e., "possible worlds"). It may seem that probabilistic data can be thought of defining a weighted instance of the deterministic problem, but we show empirically and analytically that this yields poor representations. Thus, building effective histogram or wavelet synopses for such collections of possible-world data distributions is a challenging problem that mandates novel analyses and algorithmic approaches. As in the large body of work on synopses for deterministic data, we consider a number of standard error functions, and show how to find the optimal synopsis relative to this class of function.

Our Contributions. In this paper we give the first formal definitions and algorithmic solutions for constructing histogram and wavelet synopses over probabilistic data. In particular:

- We define the "probabilistic data reduction problem" over a variety of cumulative and maximum error objectives, based on natural generalizations of histograms and wavelets from deterministic to probabilistic data.
- We give efficient techniques to find optimal histograms under all common cumulative error objectives (such as sum-squared error and sum-relative error), and corresponding maximum error objectives. We also show that fast approximate solutions are possible. Our techniques rely on careful analysis of each objective function in turn, and showing that the cost of a given bucket, along with its optimal representative value, can be found efficiently from appropriate precomputed arrays, taking into account the distributions of items, and correlations between them.
- For wavelets, we similarly show optimal techniques for the core sum-squared error (SSE) objective. Here, it suffices to compute the wavelet transformation of a deterministic input derived from the probabilistic input. We also show how to extend algorithms from the deterministic setting to probabilistic data for non-SSE objectives.
- We report on experimental evaluation of our methods, and show that they achieve appreciably better results than simple heuristics. The space and time costs are always equal or better than the heuristics.

A. Prior Work on Uncertain Data

Key ideas in probabilistic databases are presented in tutorials by Dalvi and Suciu [10], [11], and built on by systems such as Trio [4], MCDB [12] and MayBMS [3]. Initial research has focused on how to store and process uncertain data within database systems, and hence how to answer SQL-style queries. Subsequently, there has been a growing realization that in addition to storing and processing uncertain data, there is a need to run advanced algorithms to analyze uncertain data. Recent work has studied how to compute properties of streams of uncertain tuples such as the expected average and number of distinct items [13], [14], [15]; clustering uncertain data [16]; and finding frequent items within uncertain data [17].

There has been work on finding quantiles of unidimensional data [13], [15], which can be thought of as the *equi-depth* histogram; the techniques to find these show that it simplifies to the problem of finding quantiles over weighted data, where the weight of each item is simply its expected frequency. Similarly, finding frequent items is somewhat related to finding high-biased histograms. Lastly, we can also think of building a histogram as being a kind of clustering of the data along the domain. However, the nature of the error objectives on histograms that are induced by the formalizations of clustering are quite different from here, and so probabilistic clustering techniques [16] do not give good solutions for histograms. To our knowledge, no prior work studied problems of building histogram and wavelet synopses of probabilistic data. In parallel to this work, Zheng [18] studied the SSE histogram problem in the streaming context, and showed similar results to those given in Section III-A.

II. Preliminaries and Problem Formulation

A. Probabilistic Data Models

A variety of models of probabilistic data have been proposed. The different models capture various levels of independence between the individual data values described. Each model describes a distribution over *possible worlds*: each possible world is a (traditional) relation containing some number of tuples. The most general model describes the complete correlations between all tuples; effectively, it describes every possible world and its associated probability explicitly. However, the size of such a model for even a moderate number of tuples is immense, since the exponentially many possible combinations of values are spelled out. In practice, finding the (exponentially many) parameters for the fully general model is unfeasible; instead, more compact models are adopted which can reduce the number of parameters by making independence assumptions between tuples.

The simplest probabilistic model is the *basic* model, which consists of a sequence of tuples containing a single value and the probability that it exists in the data. More formally,

Definition 1: The *basic model* consists of a set of m tuples where the jth tuple consists of a pair $\langle t_j, p_j \rangle$. Here, t_j is an item drawn from a fixed domain, and p_j is the probability that t_j appears in any possible world. Each possible world W

is formed by the inclusion of a subset of the items t_j. We write $j \in W$ if t_j is present in the possible world W, and $j \notin W$ otherwise. Each tuple is assumed to be independent of all others, so the probability of possible world W is given by $\Pr[W] = \prod_{j \in W} p_j \prod_{j \notin W}(1 - p_j)$. ∎

Note here that the items t_j can be somewhat complex (e.g., a row in a table), but without loss of generality we will treat them as simple objects. In particular, we will be interested in cases that can be modeled as when the t_js are drawn from a fixed, ordered domain (such as $1, 2, 3 \ldots$) of size n, and several t_js can correspond to occurrences of the same item. We consider two extensions of the basic model, which each capture dependencies not expressible in the basic model by providing a compact discrete probability density function (pdf) in place of the single probability.

Definition 2: In the *tuple pdf* model, instead of a single (item, probability) pair, there is a set of pairs with probabilities summing to at most 1. That is, the input consists of a sequence of tuples $t_j \in \mathcal{T}$ of the form $\langle (t_{j1}, p_{j1}), \ldots (t_{j\ell}, p_{j\ell}) \rangle$. Each tuple specifies a set of mutually exclusive possible values for the ith row of a relation. The sum of the probabilities within a tuple is at most 1; the remaining probability measures the chance that there is no corresponding item. We interpret this as a discrete pdf for the jth item in the input: $\Pr[t_j = t_{j1}] = p_{j1}$, and so on. Each tuple is assumed to be independent of all others, so the probability of any possible world can be computed via multiplication of the relevant probabilities. ∎

This model has been widely adopted, and is used in Trio (i.e., OR-tuples) [4], and elsewhere [15]. It captures the case where an observer makes readings, and has some uncertainty over what was seen. An alternate case is when an observer makes readings of a known item (for example, this could be a sensor making discrete readings), but has uncertainty over a value or frequency associated with the item:

Definition 3: The *value pdf* model consists of a sequence of tuples of the form $\langle i : (f_{i1}, p_{i1}) \ldots (f_{i\ell}, p_{i\ell}) \rangle$, where the probabilities in each tuple sum to at most 1. Each tuple specifies the distribution of frequencies of a separate item; the distributions of different items are assumed to be distinct. This describes a discrete pdf for the random variable g_i giving (say) the distribution of the frequencies of the ith item: $\Pr[g_i = f_{i1}] = p_{i1}$, and so on. Due to the independence, the probability of any possible world is computed via multiplication of probabilities for the frequency of each item in turn. If probabilities in a tuple sum to less than one, the remainder is taken to implicitly specify the probability that the frequency is zero, by analogy with the basic model. Let the set of all values of frequencies used be $\mathcal{V}$, so every (f, p) pair has $f \in \mathcal{V}$. ∎

One can go on to describe more complex models which further describe the interactions between different items (e.g., via graphical models), but these are complex to instantiate in practice; prior work on query processing and mining mostly works in the models described here, or variations thereof. For both the basic and tuple models, the frequency of any given item i within a possible world, g_i, is a non-negative integer, and each occurrence corresponds to a tuple from the input. The value pdf model can specify arbitrary fractional frequencies, but the number of such frequencies is bounded by the size of the input, m. The basic model is a special case of the tuple pdf and value pdf model, but neither of these two is contained within the other. However, input in the tuple pdf model *induces* a distribution over frequencies of each item. We define the *induced value pdf* which provides $\Pr[g_i = v]$ for some $v \in \mathcal{V}$ and for each item i. The important detail is that, unlike in the value pdf model, these induced pdfs are *not* independent; nevertheless, this representation is useful in our subsequent analysis. For data presented in the tuple pdf format, building the induced value pdf for each value inductively takes time $O(|\mathcal{V}|)$ to update the partial value pdf with each new tuple. The total space required is $O(m)$.

Definition 4: Given an input in any of these models, let $\mathcal{W}$ denote the space of all possible worlds, and $\Pr[W]$ denote the probability associated with possible world $W \in \mathcal{W}$. We can then compute the *expectation* of various quantities over possible worlds as, given a function f which can be evaluated on a possible world W,

$$\mathsf{E}_{\mathcal{W}}[f] = \sum_{W \in \mathcal{W}} \Pr[W] f(W) \qquad (1)$$

Example 1: Consider the ordered domain containing the three items $1, 2, 3$. The input $\langle 1, \frac{1}{2} \rangle, \langle 2, \frac{1}{3} \rangle, \langle 2, \frac{1}{4} \rangle, \langle 3, \frac{1}{2} \rangle$ in the basic model defines the following twelve possible worlds and corresponding probabilities:

W	$\emptyset$	1	12	122	123	1223	13	2	22	23	223	3
$\Pr[W]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{5}{48}$	$\frac{1}{48}$	$\frac{5}{48}$	$\frac{1}{48}$	$\frac{1}{8}$	$\frac{5}{48}$	$\frac{1}{48}$	$\frac{5}{48}$	$\frac{1}{48}$	$\frac{1}{8}$

The input $\langle (1, \frac{1}{2}), (2, \frac{1}{3}) \rangle, \langle (2, \frac{1}{4}), (3, \frac{1}{2}) \rangle$ in the tuple pdf model defines eight possible worlds:

W	$\emptyset$	1	2	3	12	13	22	23
$\Pr[W]$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$

The input $\langle 1 : (1, \frac{1}{2}) \rangle, \langle 2 : (1, \frac{1}{3}), (2, \frac{1}{4}) \rangle, \langle 3 : (1, \frac{1}{2}) \rangle$ in the value pdf model defines the pdfs

$$\Pr[g_1 = 0] = \tfrac{1}{2}, \Pr[g_1 = 1] = \tfrac{1}{2}$$
$$\Pr[g_2 = 0] = \tfrac{5}{12}, \Pr[g_2 = 1] = \tfrac{1}{3}, \Pr[g_2 = 2] = \tfrac{1}{4}$$
$$\Pr[g_3 = 0] = \tfrac{1}{2}, \Pr[g_3 = 1] = \tfrac{1}{2}$$

and defines twelve possible worlds:

W	$\emptyset$	1	12	122	123	1223	13	2	22	23	223	3
$\Pr[W]$	$\frac{5}{48}$	$\frac{5}{48}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{5}{48}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{5}{48}$

In all three cases, $\mathsf{E}_{\mathcal{W}}[g_1] = \frac{1}{2}$. In the value pdf case, $\mathsf{E}_{\mathcal{W}}[g_2] = \frac{5}{6}$, for the other two cases $\mathsf{E}_{\mathcal{W}}[g_2] = \frac{7}{12}$. ∎

Although two possible worlds may be formed in different ways, they may be indistinguishable: for instance, in the basic model example, the world $W = \{2\}$ can result either from the second or third tuple. We will typically not distinguish possible worlds based on how they arose, and so treat them as identical. Our input is then characterized by parameters n, giving the size of the ordered domain from which the input is drawn, m, the total number of pairs in the input (hence the input can be described with $O(m)$ pieces of information), and $\mathcal{V}$, the set of values that the frequencies take on. Here $|\mathcal{V}| \leq m$, but could be much less. In the all three examples above, we have $n = 3, m = 4$, and $\mathcal{V} = \{0, 1, 2\}$.

B. Histogram and Wavelet Synopses

Histograms on Deterministic Data. Consider a one-dimensional data distribution defined (without loss of generality) over the integer domain $[n] = \{0, \ldots, n-1\}$. For each $i \in [n]$, let g_i denote the *frequency* of domain value i in the underlying data set. A *histogram* synopsis provides a concise, piece-wise approximate representation of the distribution based on partitioning the ordered domain $[n]$ into B *buckets*. Each bucket b_k consists of a *start and end point*, $b_k = (s_k, e_k)$, and approximates the frequencies of the contiguous subsequence of values $\{s_k, s_k+1, \ldots, e_k\}$ (termed the *span* of the bucket) using a single representative value $\hat{b}_k$. Let $n_k = e_k - s_k + 1$ denote the width (i.e., number of distinct items) of bucket b_k. The B buckets in a histogram form a *partition* of $[n]$; that is, $s_1 = 0$, $e_B = n-1$, and $s_{k+1} = e_k + 1$ for all $k = 1, \ldots, B-1$.

By using $\mathrm{O}(B) \ll n$ space to represent an $O(n)$-size data distribution, histograms provide a very effective means of data reduction, with numerous applications [6]. This data reduction implies *approximation errors* in the estimation of frequencies, since each $g_i \in b_k$ is estimated as $\hat{g}_i = \hat{b}_k$. The histogram construction problem is, given a storage budget B, build a B-bucket histogram $\mathcal{H}_B$ that is *optimal under some aggregate error metric*. The histogram error metrics to minimize include, for instance, the *Sum-Squared-Error*

$$\mathrm{SSE}(\mathcal{H}) = \textstyle\sum_{i=1}^{n} (g_i - \hat{g}_i)^2 = \sum_{k=1}^{B} \sum_{i=s_k}^{e_k} (g_i - \hat{b}_k)^2$$

(which defines the important class of *V-optimal* histograms [9], [19]) and the *Sum-Squared-Relative-Error*:

$$\mathrm{SSRE}(\mathcal{H}) = \textstyle\sum_{i=1}^{n} \frac{(g_i - \hat{g}_i)^2}{\max\{c, |g_i|\}^2}$$

(where constant c in the denominator avoids excessive emphasis being placed on small frequencies [20], [21]). The *Sum-Absolute-Error* (SAE) and *Sum-Absolute-Relative-Error* (SARE) are defined similarly to SSE and SSRE, replacing the square with an absolute value so that

$$\mathrm{SAE}(\mathcal{H}) = \sum_{i=1}^{n} |g_i - \hat{g}_i| \ ; \ \mathrm{SARE}(\mathcal{H}) = \sum_{i=1}^{n} \frac{|g_i - \hat{g}_i|}{\max\{c, |g_i|\}}$$

In addition to such *cumulative* metrics, *maximum error metrics* provide approximation guarantees on the relative/absolute error of individual frequency approximations [21]; these include *Maximum-Absolute-Relative-Error* $\mathrm{MARE}(\mathcal{H}) = \max_{i \in [n]} \frac{|g_i - \hat{g}_i|}{\max\{c, |g_i|\}}$.

Histogram construction satisfies the *principle of optimality:* If the B^{th} bucket in the optimal histogram spans the range $[i+1, n-1]$, then the remaining $B-1$ buckets must form an optimal histogram for the range $[0, i]$. This immediately leads to a Dynamic-Programming (DP) algorithm for computing the optimal error value $\textsc{opth}[j, b]$ for a b-bucket histogram spanning the prefix $[1, j]$ based on the following recurrence:

$$\textsc{opth}[j, b] = \min_{0 \le l < j} \{h(\textsc{opth}[l, b-1], \min_{\hat{b}} \{\textsc{be}([l+1, j], \hat{b})\})\}, \quad (2)$$

where $\textsc{be}([x, y], z)$ denotes the error contribution of a *single* histogram bucket spanning $[x, y]$ using a representative value of z to approximate all enclosed frequencies, and $h(x, y)$ is simply $x + y$ (respectively, $\max\{x, y\}$) for cumulative (resp., maximum) error objectives. The key to translating the above recurrence into a fast algorithm lies in being able to quickly find the best representative $\hat{b}$ and the corresponding optimal error value $\textsc{be}()$ for the *single-bucket case*.

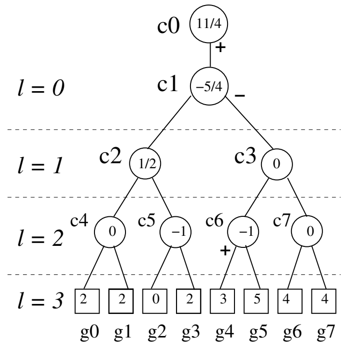


Fig. 1. Error-tree structure on $n = 8$ items for data distribution array $A = [2, 2, 0, 2, 3, 5, 4, 4]$ (l = coefficient resolution levels.)

Wavelets on Deterministic Data. *Haar wavelet synopses* [8], [22], [23], [24] provide another data reduction tool based on the Haar Discrete Wavelet Decomposition (DWT) [25] for hierarchically decomposing functions. At a high level, the Haar DWT of a data distribution over $[n]$ consists of a coarse overall approximation (the average of all frequencies) together with $n-1$ *detail coefficients* (constructed through recursive pairwise averaging and differencing) that influence the reconstruction of frequency values at different scales. The Haar DWT process can be visualized through a binary *coefficient tree* structure, as in Figure 1. Leaf nodes g_i correspond to the original data distribution values in $A[]$. The root node c_0 is the overall average frequency, whereas each internal node c_i ($i = 1, \ldots, 7$) is a detail coefficient computed as the half the difference between the average of frequencies in c_i's left child subtree and the average of frequencies in c_i's right child subtree (e.g., $c_3 = \frac{1}{2}(\frac{3+5}{2} - \frac{4+4}{2}) = 0$). Coefficients in level l are normalized by a factor of $\sqrt{2^l}$: this makes the transform an *orthonormal basis* [25], which preserves the L_2 norm.

Any data value g_i can be reconstructed as a function of the coefficients which are proper ancestors of the corresponding node in the coefficient tree: the reconstructed value can be found by summing appropriately scaled multiples of these $\log N + 1$ coefficients alone. The *support* of a coefficient c_i is defined as the interval of data values that c_i is used to reconstruct; it is a dyadic interval of size $2^{\log n - l}$ for a coefficient at *resolution level* l (see Fig. 1).

Given limited space for maintaining a *wavelet synopsis* a *thresholding procedure* retains $B \ll n$ Haar coefficients as a highly-compressed approximate representation of the data (remaining coefficients are implicitly set to 0). As in to histogram construction, the aim is to determine the "best" subset of B coefficients to retain, so that some overall error measure is minimized. By orthonormality of the normalized

Haar basis, greedily picking the B (absolute, normalized) largest coefficients (based on *absolute normalized value*) is optimal for SSE error [25]. Recent work proposes schemes for optimal and approximate thresholding under different error metrics. These schemes formulate a dynamic program over the coefficient-tree structure that tabulates the optimal solution for a subtree rooted at node c_j *given the contribution from the choices made at proper-ancestor nodes of* c_j *in the tree*. This handles a broad class of *distributive error metrics* (including all error measures discussed above) [22], [23].

There are two distinct versions of the thresholding problem for non-SSE error metrics. In the *restricted* version the thresholding algorithm is forced to select values for the synopsis from the standard Haar coefficient values (computed as discussed above), which can lead to sub-optimal synopses for non-SSE error [23]. In the *unrestricted* version of the problem, retained coefficient values are chosen to optimize the target error metric [23]. Let $\textsc{optw}[j,b,v]$ denote the optimal error contribution across all frequencies g_i in the support (i.e., subtree) of coefficient c_j assuming a total space budget of b coefficients retained in c_j's subtree; and, a (partial) reconstructed value of v based on the choices made at proper ancestors of c_j. Then, based on the Haar DWT reconstruction process, we can compute $\textsc{optw}[j,b,v]$ as the minimum of two alternative error values at c_j:
(1) Optimal error when retaining the best value for c_j*,* found by minimizing over all values v_j for c_j and allotments of the remaining budget across the left and right child of c_j, i.e.,

$$\textsc{optw}_r[j,b,v] = \min_{v_j, 0\le b'\le b-1} \{h(\textsc{optw}[2j,b',v+v_j], \textsc{optw}[2j+1,b-b'-1,v-v_j])\}.$$

(2) Optimal error when not retaining c_j*,* computed similarly:

$$\textsc{optw}_{nr}[j,b,v] = \min_{v_j, 0\le b'\le b} \{h(\textsc{optw}[2j,b',v], \textsc{optw}[2j+1,b-b',v])\}$$

where $h()$ stands for summation ($\max\{\}$) for cumulative (resp., maximum) error-metric objectives. In the *restricted* problem, minimization over v_j is eliminated (since the value for c_j is fixed), and the values for the "incoming" contribution v can be computed by stepping through *all possible subsets of ancestors* for c_j — since the depth of the tree is $O(\log n)$, this implies an $O(n^2)$ thresholding algorithm [22]. In the *unrestricted* case, Guha and Harb propose efficient approximation schemes that employ techniques for bounding and approximating the range of possible v values [23].

C. Probabilistic Data Reduction Problem

The key difference in the probabilistic data setting is that data-distribution frequencies g_i (and the Haar DWT coefficients) are now *random variables*. We use $g_i(W)$ ($c_i(W)$) to denote the (instantiated) frequency of the i^{th} item (resp., value of the i^{th} Haar coefficient) in possible world W— we can omit the explicit dependence on W for the sake of conciseness. The error of a given data synopsis is also a random variable over possible worlds. Our goal becomes that of constructing a data synopsis (histogram or wavelet) that optimizes *an expected measure of the target error objective over possible worlds*. More formally, let $\texttt{err}(g_i,\hat{g}_i)$ denote the error of approximating g_i by $\hat{g}_i$ (e.g., squared error or absolute relative error for item i); then, our problem can be formulated as follows.

[Synopsis Construction for Probabilistic Data] Given a collection of probabilistic attribute values, a synopsis space budget B, and a target (cumulative or maximum) error metric, determine a size-B synopsis that minimizes either (1) the expected cumulative error over all possible worlds, i.e., $\mathsf{E}_{\mathcal{W}}[\sum_i \texttt{err}(g_i,\hat{g}_i)]$ (in the case of a cumulative error objective); or, (2) the maximum value of the per-item expected error over all possible worlds, i.e., $\max_i\{\mathsf{E}_{\mathcal{W}}[\texttt{err}(g_i,\hat{g}_i)]\}$ (for a maximum error objective). ∎

A natural first attempt to solve the probabilistic data reduction problem is to try sampling, or building a weighted deterministic data set. More precisely, one could try the Monte-Carlo approach of sampling a possible world W with probability $\Pr[W]$ and building the optimal synopsis for W (similar to [12]); or for each item i finding $\mathsf{E}_{\mathcal{W}}[g_i]$, and building the synopsis of the "expected" data. Our subsequent analysis shows that such attempts are insufficient. We give precise formulations of the optimal solution to the problem under a variety of error metrics, and one can verify by inspection that they do not in general correspond to any of these simple solutions. Further, we compare the optimal solution to these solutions in our experimental evaluation, and observe that the quality of the solution found is substantially poorer. This stands in contrast to prior work on estimating functions such as expected number of distinct items [13], [15], which analyze the number of samples needed to give an accurate estimate. This is largely because synopses are not scalar values—it is not meaningful to find the "average" of multiple synopses.

III. Histograms on Uncertain Data

We first consider optimal histograms under cumulative error objectives which minimize the expected cost of the histogram. Our techniques are based on applying the dynamic programming approach. Most of our effort is in showing how to compute the optimal $\hat{b}$ for a bucket b under a given error objective, and also to compute the corresponding value of $\mathsf{E}_{\mathcal{W}}[\textsc{be}(b,\hat{b})]$. Here, we observe that the principle of optimality still holds even under uncertain data: since the expectation of the sum of costs of each bucket is equal to the sum of the expectations, removing the final bucket should leave an optimal $B-1$ bucket histogram over the prefix of the domain. Hence, we will be able to invoke equation (2), and find a solution which evaluates $O(Bn^2)$ possibilities.

A. Sum-squared error Histograms

SSE is the sum of the squared differences between the values within a bucket b_k and the representative value of the bucket, $\hat{b}_k$. For a fixed possible world W, the optimal value for $\hat{b}_i$ is the mean of the frequencies g_i in the bucket and the measure reduces to a multiple of the sample variance of the values within the bucket. This holds true even under probabilistic data, as we show below. For deterministic data, it is straightforward to quickly compute the (sample) variance within a given bucket, and therefore use dynamic programming to select the optimum bucketing [19]. To use the DP approach on uncertain data, which specifies exponentially many possible worlds, we must efficiently compute the variance in a given bucket b specified by start point s and end point e.

Given the (fixed) frequency distribution implied by W, and bucket b of span n_b, the sample variance of W, $\sigma_b^2(W)$, is defined from $g_i(W)$ (frequency of item i in world W), as

$$\sigma_b^2(W) = \sum_{i=s}^{e} \frac{(g_i - \bar{b})^2}{n_b} = \left(\sum_{i=s}^{e} \frac{g_i^2}{n_b}\right) - \left(\sum_{i=s}^{e} \frac{g_i}{n_b}\right)^2 \quad (3)$$

Given a distribution over $\mathcal{W}$ where $\Pr[W]$, the probability of possible world W, substituting the variance in (1) gives

$$\mathsf{Var}_{\mathcal{W}}(b) = \mathsf{E}_{\mathcal{W}}[\sigma_b^2] = \sum_{W \in \mathcal{W}} \sigma_b^2(W) \Pr[W] \quad (4)$$

Fact 1: Under the sum-squared error measure, the cost is minimized by setting $\hat{b} = \mathsf{E}_{\mathcal{W}}[\sum_{i=s}^{e} g_i] = \bar{b}$.

This follows from simple algebra, since the cost of the bucket can be shown to be $n_b(\mathsf{Var}_{\mathcal{W}}(b) + (\hat{b} - \bar{b})^2)$. Using equations (3) and (4), we can write $\mathrm{SSE}(b, \bar{b}) = n_b \mathsf{Var}_{\mathcal{W}}(b)$ as the combination of two terms:

$$\begin{aligned}\mathrm{SSE}(b,\bar{b}) &= \sum_{W\in\mathcal{W}} \Pr[W] \sum_{i=s}^{e} g_i(W)^2 - \sum_{W\in\mathcal{W}} \frac{\Pr[W]}{n_b} (\sum_{i=s}^{e} g_i(W))^2 \\ &= \sum_{i=s}^{e} \mathsf{E}_{\mathcal{W}}[g_i^2] - \frac{1}{n_b} \mathsf{E}_{\mathcal{W}}[(\sum_{i=s}^{e} g_i)^2] \quad (5)\end{aligned}$$

The first term is the expectation (over possible worlds) of the sum of squares of frequencies of each item in the bucket. The second term is the expected square of the weight of the bucket, scaled by the span of the bucket. We show how to compute each term efficiently in our different models.

Value pdf model. In the value pdf model, we have a distribution for each item i over frequency values $v_j \in \mathcal{V}$ giving $\Pr[g_i = v_j]$. By independence of the pdfs, we have

$$\sum_{i=s}^{e} \mathsf{E}_{\mathcal{W}}[g_i^2] = \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] v_j^2$$

Since for any random variable X, $\mathsf{E}[X^2] = \mathsf{Var}[X] + \mathsf{E}[X]^2$, by linearity of expectation and summation of variance the second term in equation (5) is

$$\begin{aligned}\mathsf{E}_{\mathcal{W}}[(\sum_{i=s}^{e} g_i)^2] &= \mathsf{E}_{\mathcal{W}}[\sum_{i=s}^{e} g_i]^2 + \mathsf{Var}_{\mathcal{W}}[\sum_{i=s}^{e} g_i] \quad (6)\\ &= (\sum_{i=s}^{e} \sum_{v_j\in\mathcal{V}} \Pr[g_i = v_j] v_j)^2 + \sum_{i=s}^{e} \mathsf{Var}_{\mathcal{W}}[g_i]\end{aligned}$$

where, in turn

$$\mathsf{Var}_{\mathcal{W}}[g_i] = \sum_{v_j\in\mathcal{V}} \Pr[g_i = v_j] v_j^2 - (\sum_{v_j\in\mathcal{V}} \Pr[g_i = v_j] v_j)^2$$

Tuple pdf model. In the tuple pdf case, things seem more involved, due to interactions between items in the same tuple. As shown by equation (5), we need to compute $\mathsf{E}_{\mathcal{W}}[g_i^2]$ and $\mathsf{E}_{\mathcal{W}}[(\sum_i g_i)^2]$. Let the set of tuples in the input be $\mathcal{T} = \{t_j\}$, so each tuple has an associated pdf giving $\Pr[t_j = i]$, from which is derived $\Pr[a \le t_j \le b]$, the probability that the ith tuple in the input is falls between a and b in the input domain.

$$\begin{aligned}\mathsf{E}_{\mathcal{W}}[g_i^2] &= \mathsf{Var}_{\mathcal{W}}[g_i] + (\mathsf{E}_{\mathcal{W}}[g_i])^2 \\ &= \sum_{t_j\in\mathcal{T}} \Pr[t_j = i](1 - \Pr[t_j = i]) + (\sum_{t_j\in\mathcal{T}} \Pr[t_j = i])^2\end{aligned}$$

Here, we rely on the fact that variance of each g_i is the sum of the variances arising from each tuple in the input. Observe that although there are dependencies between particular items, these do not affect the computation of the expectations for individual items, which can be then be summed to find the overall answer. For the second term in (5), we can use the expression (6), and we already have an expression for $(\mathsf{E}_{\mathcal{W}}[\sum_i g_i])^2$. But we cannot simply write $\mathsf{Var}_{\mathcal{W}}[\sum_{i=s}^{e} g_i]$ as the sum of variances, since these are no longer independent variables. So instead, we treat all items in the same bucket together as a single item and compute its expected square by iterating over all tuples in the input $\mathcal{T}$:

$$\mathsf{Var}_{\mathcal{W}}[\textstyle\sum_{i=s}^{e} g_i] = \sum_{t_j\in\mathcal{T}} \Pr[s \le t_j \le e](1 - \Pr[s \le t_j \le e])$$

Efficient Computation. The above development shows how to find the cost of a specified bucket. Computing the minimum cost histogram requires comparing the cost of many different choices of buckets. As in the deterministic case, since the cost is the sum of the costs of the buckets, the dynamic programming solution can find the optimal cost. This computes the cost of the optimal j bucket solution up to position ℓ combined with the cost of the optimal $k - j$ bucket solution over positions $\ell + 1$ to n. This means finding the cost of $O(n^2)$ buckets. By analyzing the form of the above expressions for the cost of a bucket, we can precompute enough information to allow the cost of any specified bucket to be found in time $O(1)$.

For the tuple pdf model (the value pdf model is similar but simpler) we precompute arrays of length n:

$$A[e] = \sum_{i=1}^{e} (\sum_{t_j\in\mathcal{T}} \Pr[t_j = i](1 - \Pr[t_j = i]) + (\sum_{t_j\in\mathcal{T}} \Pr[t_j = i])^2)$$

$$B[e] = \sum_{t_j\in\mathcal{T}} \Pr[t_j \le e] \qquad C[e] = \sum_{t_j\in\mathcal{T}} (\Pr[t_j \le e])^2$$

for $1 \le e \le s$, and set $A[0] = B[0] = C[0] = 0$. Then the cost $\mathrm{SSE}((s,e), \bar{b})$ is (after symbolic manipulation) given by

$$A[e] - A[s-1] - \frac{(B[e]-B[s-1])(B[e]+B[s-1]+1) - (C[e]-C[s-1])}{(e-s+1)}$$

With the input in sorted order, these three arrays can be computed with a linear pass over the input. In summary:

Theorem 1: Optimal SSE histograms can be computed over probabilistic data presented in the value pdf or tuple pdf models in time $O(m + Bn^2)$.

B. Sum-Squared-Relative-Error Histograms

SSRE over deterministic data computes the difference between $\hat{b}$, the representative value for the bucket, and each value within the bucket, and reports the square of this difference as a ratio to the square of the corresponding value. An additional 'sanity' parameter c limites this ratio in case some values in the bucket are very small. With probabilistic data, we take the expected value of this quantity. The cost is

$$\mathrm{SSRE}(b,\hat{b}) = \mathsf{E}_{\mathcal{W}}[\textstyle\sum_{i=s}^{e} \frac{(g_i-\hat{b})^2}{\max(c^2,g_i^2)}]$$

By linearity of expectation, the cost given $\hat{b}$ can be computed by evaluating all frequencies which have a non-zero probability (i.e. at all $v \in \mathcal{V}$).

Value pdf model. We write the SSRE cost in terms of the probability that, over all possible worlds, the ith item has frequency $v_j \in \mathcal{V}$. Then

$$\mathrm{SSRE}(b,\hat{b}) = \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] \frac{(v_j - \hat{b})^2}{\max(c^2, v_j^2)} \qquad (7)$$

We rewrite this using the function $w(x) = 1/\max(c^2,x^2)$, which is a fixed value once x is specified. Now our cost is

$$\mathrm{SSRE}(b,\hat{b}) = \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \big(\Pr[g_i = v_j] w(v_j) v_j^2 - 2\Pr[g_i = v_j] w(v_j) v_j \hat{b} + \Pr[g_i = v_j] w(v_j) \hat{b}^2 \big)$$

which is a quadratic in $\hat{b}$. Simple calculus demonstrates that the optimal value of $\hat{b}$ to minimize this cost is

$$\hat{b} = \frac{\sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] v_j w(v_j)}{\sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] w(v_j)}.$$

Substituting this value of $\hat{b}$ gives $\mathrm{SSRE}(b,\hat{b}) =$

$$\sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] v_j^2 w(v_j) - \frac{(\sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] v_j w(v_j))^2}{\sum_{i=s}^{e} \sum_{v_j \in V} \Pr[g_i = v_j] w(v_j)}$$

Define the following arrays:

$$\begin{aligned} X[e] = & \textstyle\sum_{i=1}^{e} \sum_{v_j \in V} \Pr[g_i = v_j] v_j^2 w(v_j) \\ Y[e] = & \textstyle\sum_{i=1}^{e} \sum_{v_j \in V} \Pr[g_i = v_j] v_j w(v_j) \\ Z[e] = & \textstyle\sum_{i=1}^{e} \sum_{v_j \in V} \Pr[g_i = v_j] w(v_j) \end{aligned}$$

The cost of any bucket is found in constant time:

$$\min_{\hat{b}} \mathrm{SSE}((s,e),\hat{b}) = X[e] - X[s-1] - \frac{(Y[e]-Y[s-1])^2}{Z[e]-Z[s-1]}$$

Dynamic programming then finds the optimal set of buckets.

Tuple pdf model. For this cost measure, the cost for the bucket b given by equation (7) is the sum of costs obtained by each item in the bucket. We can focus solely on the contribution to the cost made by a single item i, and observe that equation (7) depends only on the (induced) distribution giving $\Pr[g_i = v_j]$: there is no dependency on any other item. So we compute the induced value pdf (Section II-A) for each item independently, and apply the above analysis.

Theorem 2: Optimal SSRE histograms be can computed over probabilistic data presented in the value pdf model in time $O(m+Bn^2)$ and $O(m|\mathcal{V}|+Bn^2)$ in the tuple pdf model.

C. Sum-Absolute-Error Histograms

Let $\mathcal{V}$ be the set of possible values taken on by the g_is, indexed so that $v_1 \le v_2 \le \ldots \le v_{|\mathcal{V}|}$. Given some $\hat{b}$, let j' satisfy $v_{j'} \le \hat{b} < v_{j'+1}$ (we can insert 'dummy' values of $v_0 = 0$ and $v_{|\mathcal{V}|+1} = \infty$ if $\hat{b}$ falls outside of $v_1 \ldots v_{|\mathcal{V}|}$). The sum of absolute errors is given by

$$\begin{aligned} \mathrm{SAE}(b,\hat{b}) &= \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] |\hat{b} - v_j| \\ &= \sum_{i=s}^{e} (\hat{b} - v_{j'}) \Pr[g_i \le v_{j'}] + (v_{j'+1} - \hat{b}) \Pr[g_i \ge v_{j'+1}] \\ &\quad + \sum_{v_j \in \mathcal{V}} \begin{cases} \Pr[g_i \le v_j](v_{j+1} - v_j) & \text{if } v_j < v_{j'} \\ \Pr[g_i > v_j](v_{j+1} - v_j) & \text{if } v_j \ge v_{j'} \end{cases} \end{aligned}$$

The contribution of the first two terms can be written as

$$\begin{aligned} (v_{j'+1} - v_{j'}) \quad & \Pr[g_i \le v_{j'}] + \\ (\hat{b} - v_{j'+1}) \quad & (\Pr[g_i \le v_{j'}] - \Pr[g_i \ge v_{j'+1}]) \end{aligned}$$

This gives a quantity that is independent of $\hat{b}$ added to one that depends linearly on $\Pr[g_i \le v_{j'}] - \Pr[g_i > v_{j'+1}]$, which we define as $\Delta_{j'}$. So if $\Delta_{j'} > 0$, we can reduce the cost by making $\hat{b}$ closer to $v_{j'+1}$; if $\Delta_{j'} < 0$, we can reduce the cost by making $\hat{b}$ closer to $v_{j'}$. Therefore, the optimal value of $\hat{b}$ occurs when we make it equal to some v_j value (since, when $\Delta_{j'} = 0$, we have the same result when setting $\hat{b}$ to either $v_{j'}$ or $v_{j'+1}$, or anywhere in between). So, we assume that $\hat{b} = v_{j'}$ for some $v_{j'} \in \mathcal{V}$ and can state

$$\mathrm{SAE}(b,\hat{b}) = \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \begin{cases} \Pr[g_i \le v_j](v_{j+1} - v_j) & \text{if } \hat{b} > v_j \\ \Pr[g_i > v_j](v_{j+1} - v_j) & \text{if } \hat{b} \le v_j \end{cases}$$

Set $P_{j,s,e} = \sum_{i=s}^{e} \Pr[g_i \le v_j]$ and $P^*_{j,s,e} = \sum_{i=s}^{e} \Pr[g_i > v_j]$. Observe that $P_{j,s,e}$ is monotone increasing in j while $P^*_{j,s,e}$ is monotone decreasing in j. So we have

$$\mathrm{SAE}(b,\hat{b}) = \sum_{v_j < \hat{b}} P_{j,s,e}(v_{j+1} - v_j) + \sum_{v_j \ge \hat{b}} (P^*_{j,s,e})(v_{j+1} - v_j) \qquad (8)$$

There is a contribution of $(v_{j+1} - v_j)$ for all j values, multiplied by either $P[j,s,e]$ or $P^*[j,s,e]$. Consider the effect on SAE if we step $\hat{b}$ through values $v_1, v_2 \ldots v_{|\mathcal{V}|}$. We have

$$\mathrm{SAE}(b, v_{\ell+1}) - \mathrm{SAE}(b, v_\ell) = (P_{\ell,s,e} - P^*_{\ell+1,s,e})(v_{\ell+1} - v_\ell)$$

Because $P_{j,s,e}$ is monotone increasing in j, and $P^*_{j,s,e}$ is monotone decreasing in j, the quantity $P_{\ell,s,e} - P^*_{\ell+1,s,e}$ is monotone increasing in ℓ. Thus $\mathrm{SAE}(b,\hat{b})$ can have a single minimum value as $\hat{b}$ is varied, and is increasing in both directions away from this value. This minimum does not depend on the v_j values themselves; instead, it occurs (approximately) when

$P_{j,s,e} \approx P^*_{j,s,e} \approx n_b/2$, where $n_b = (e-s+1)$ as before. It therefore suffices to find the v'_j value defined by

$$v'_j = \arg\min_{v_\ell \in \mathcal{V}} \sum_{v_j < v_\ell \in \mathcal{V}} P_{j,s,e} + \sum_{v_j > v_\ell \in \mathcal{V}} P^*_{j,s,e}$$

and then set $\hat{b} = v_{j'}$ to obtain the optimal SAE cost.

Tuple and Value pdf models. In the value pdf case, it is straightforward to compute the P and P^* values directly from the input pdfs. For the tuple pdf case, observe that from the form of the expression for SAE, there are no interactions *between* different g_i values: although the input specifies interactions and (anti)correlations between different variables, for computing the error in a bucket we can treat each item independently in turn. We therefore convert to the induced value pdf (at an additional cost of $O(m|\mathcal{V}|)$), and use this in our subsequent computations.

Efficient Computation. To quickly find the cost of a given bucket b, we first find the optimal $\hat{b}$. We precompute $\sum_{v_j < \ell} P_{j,1,e}(v_{j+1} - v_j)$ and $\sum_{v_j \geq \ell} P^*_{j,1,e}(v_{j+1} - v_j)$ values for all $v_\ell \in \mathcal{V}$ and $e \in [n]$. Now $\mathrm{SAE}(b, \hat{b})$ for any $\hat{b} \in \mathcal{V}$ can be computed (from (8)) as the sum of two differences of precomputed values. The minimum value attainable by any $\hat{b}$ can then be found by a ternary search over the values $\mathcal{V}$, using $O(\log|\mathcal{V}|)$ probes. The time cost is $O(|\mathcal{V}|n)$ preprocessing to build tables of prefix sums, and $O(\log|\mathcal{V}|)$ to find the optimal cost of a given bucket. Therefore,

Theorem 3: Optimal SAE histograms can be computed over probabilistic data presented in the (induced) value pdf model in time $O(n(|\mathcal{V}| + Bn + n\log|\mathcal{V}|))$.

For all of models of probabilistic data, $|\mathcal{V}| \leq m$ is polynomial in the size of the input, so the total cost is polynomial.

D. Sum-Absolute-Relative-Error Histograms

For sum of absolute relative errors, bucket cost $\mathrm{SARE}(b, \hat{b})$ is

$$\begin{aligned} \mathsf{E}_{\mathcal{W}}\left(\sum_{i=s}^{e} \tfrac{|g_i - \hat{b}|}{\max(c, g_i)}\right) &= \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \tfrac{\Pr[g_i = v_j]}{\max(c, v_j)} |v_j - \hat{b}| \\ &= \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} w_{i,j} |v_j - \hat{b}| \end{aligned}$$

where we define $w_{i,j} = \frac{\Pr[g_i = v_j]}{\max(c, v_j)}$. But, more generally, the $w_{i,j}$ can be arbitrary non-negative weights. Setting j' so that $v_{j'} \leq \hat{b} \leq v_{j'+1}$, we can write the cost as

$$\sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \begin{cases} w_{i,j}(\hat{b} - v_j) & \text{if } v_j < \hat{b} \\ w_{i,j}(v_j - \hat{b}) & \text{if } v_j \geq \hat{b} \end{cases}$$

$$= \sum_{i=s}^{e} \sum_{v_j \in \mathcal{V}} \begin{cases} w_{i,j}(\hat{b} - v_{j'} + \sum_{v_j \leq v_\ell < v_{j'}} v_{\ell+1} - v_\ell) & \text{if } v_j < \hat{b} \\ w_{i,j}(v_{j'} - \hat{b} + \sum_{v_{j'} \leq v_\ell < v_j} v_{\ell+1} - v_\ell) & \text{if } v_j \geq \hat{b} \end{cases} \tag{9}$$

We define $W_{i,j} = \sum_{r=1}^{j} w_{i,r}$ and $W^*_{i,j} = \sum_{r=j+1}^{|V|} w_{i,r}$, so

$$\mathrm{SARE}(b, \hat{b}) = \sum_{i=s}^{e} W_{i,j'}(\hat{b} - v_{j'}) - W^*_{i,j}(\hat{b} - v_{j'+1})$$
$$+ \sum_{v_j \in \mathcal{V}} \begin{cases} W_{i,j}(v_{j+1} - v_j) & \text{for } v_{j'} > v_j \\ W^*_{i,j}(v_j - v_{j-1}) & \text{for } v_{j'} \leq v_j \end{cases}$$

The same style of argument as above suffices to show that the optimal choice of $\hat{b}$ is when $\hat{b} = v_{j'}$ for some j'. We define $P_{j,s,e} = \sum_{i=s}^{e} W_{i,\ell}$ and $P^*_{j,s,e} = \sum_{i=s}^{e} W^*_{i,\ell}$ so we have

$$\mathrm{SARE}(b, \hat{b}) = \sum_{v_{j'} > v_j \in V} P_{j,s,e}(v_{j+1} - v_j) + \sum_{v_{j'} \leq v_j \in V} P^*_{j,s,e}(v_{j+1} - v_j).$$

Observe that this matches the form of (8). As in Section III-C, $P_{j,s,e}$ is monotone increasing in j, and $P^*_{j,s,e}$ is decreasing in j. Therefore, the same argument holds to show that there is a unique minimum value of SARE, and it can be found by a ternary search over the range. Likewise, the form of the cost in equation (9) shows that there are no interactions between different items, so we work in the (induced) value pdf model. By building corresponding data structures based on tabulating prefix sums of the new P and P^* functions, we conclude:

Theorem 4: Optimal SARE histograms can be computed over probabilistic data presented in the tuple and value pdf models in time $O(n(|\mathcal{V}| + Bn + n\log|\mathcal{V}|\log n))$.

E. Approximate Histogram Computation

Our results so far all cost at least $\Omega(Bn^2)$ due to the use of dynamic programming to find the optimal bucket boundaries. As observed in prior work, it is not always profitable to expend so much effort when the resulting histogram only approximates original input; clearly, if we tolerate approximation in this way, then we should also be able to tolerate a histogram which achieves close to the optimal cost rather than exactly the optimal. In particular, we should be happy to find a histogram whose cost is at most $(1+\epsilon)$ times the cost of the optimal histogram in time much faster than $\Omega(Bn^2)$.

Here, we adopt the approach of Guha *et al.* [26], [27]. Instead of considering every possible bucket, we use properties of the error measure, and consider only a subset of possible buckets, much accelerating the search. We observe that the following conditions hold for all the previously considered error measures: (1) The error of a bucket only depends on the size of the bucket and the distributions of the items falling within it; (2) The overall error is the sum of the errors across all buckets; (3) We can maintain information so that given any bucket b the best representative $\hat{b}$ and corresponding error can be computed efficiently; (4) The error is monotone, so that the error for any interval of items is no less than the error of any contained subinterval; and, (5) The total error cost is bounded as a polynomial in the size of the input.

Most of our work so far has been towards point (3) above; the remainder of the points are simple consequences of the definition of the error measure. As a result of these properties, we invoke Theorem 6 of [27], and state

Theorem 5: With preprocessing as described above, we find a $(1+\epsilon)$-approximation to the optimal histogram for SSE, SSRE, SAE and SARE, with $O(\frac{1}{\epsilon}B^2 n \log n)$ bucket cost evaluations using the algorithm given in [27].

F. Max-Absolute-Error and Max-Absolute-Relative-Error

Thus far, we have relied on the linearity of expectation and related properties such as summability of variance to simplify the expressions of cost and aid in the analysis. When we consider other error metrics, such as the maximum error and maximum relative error, we cannot immediately use such linearity properties, and so the task becomes more involved. Here, we provide results for maximum absolute error and maximum absolute relative error, MAE and MARE. Over a deterministic input, the maximum error in a bucket b is $\mathrm{MAE}(b, \hat{b}) = \max_{s \le i \le e} |g_i - \hat{b}|$, and the maximum relative error is $\mathrm{MARE}(b, \hat{b}) = \max_{s \le i \le e} \frac{|g_i - \hat{b}|}{\max(c, g_i)}$. We focus on bounding the maximum value of the per-item expected error[1]. Here, we consider the frequency of each item in the bucket in turn for the expectation, and then take the maximum over these costs. So, we can write the costs as

$$\mathrm{MAE}(b, \hat{b}) = \max_{s \le i \le e} \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] |v_j - \hat{b}|$$

$$\mathrm{MARE}(b, \hat{b}) = \max_{s \le i \le e} \sum_{v_j \in \mathcal{V}} \frac{\Pr[g_i = v_j]}{\max(c, v_j)} |v_j - \hat{b}|$$

We can represent these both as $\max_{s \le i \le e} \sum_{j=1}^{|V|} w_{i,j} |v_j - \hat{b}|$, where $w_{i,j}$ are non-negative weights independent of $\hat{b}$. Now observe that we have the maximum over what can be thought of as n_b parallel instances of a sum-absolute relative error (SARE) problem, one for each i value. Following the analysis in Section III-D, we observe that each function $f_i(b) = \sum_{j=1}^{|\mathcal{V}|} |v_j - \hat{b}|$ has a single minimum value, and is increasing away from its minimum. It follows that the upper envelope of these functions, given by $\max_{s \le i \le e} f_i(b)$ also has a single minimum value, and is increasing as we move away from this minimum. So we can perform a ternary search over the values of v_j to find j' such that the optimal $\hat{b}$ lies between $v_{j'}$ and $v_{j'+1}$. Each evaluation for a chosen value of $\hat{b}$ can be completed in time $O(n_b)$: that is, $O(1)$ for each value of i, by creating the appropriate prefix sums as discussed in Section III-D (it is possible to improve this cost by appropriate precomputations, but this will not significantly alter the asymptotic cost of the whole operation). The ternary search over the values in $\mathcal{V}$ takes $O(\log |\mathcal{V}|)$ evaluations, giving a total cost of $O(n_b \log |\mathcal{V}|)$.

As $\hat{b}$ must lie in this range, the cost is of the form

$$\begin{aligned}\mathrm{MARE}(b, \hat{b}) &= \max_{s \le i \le e} \alpha_i(\hat{b} - v_{j'}) + \beta_i(v_{j'+1} - \hat{b}) + \gamma_i \\ &= \max_{s \le i \le e} \hat{b}(\alpha_i - \beta_i) + (\gamma_i + \beta_i v_{j'+1} - \alpha_i v_{j'})\end{aligned}$$

where the $\alpha_i, \beta_i, \gamma_i$ values are determined solely by j', the $w_{i,j}$'s and the v_js, and are independent of $\hat{b}$. This means we must now minimize the maximum value of a set of univariate linear functions in the range $v_{j'} \le \hat{b} \le v_{j'+1}$. A divide-and-conquer approach, based on recursively finding the intersection of convex hulls of subsets of the linear functions yields an $O(n_b \log n_b)$ time algorithm[2]. Combining these, we determine that evaluating the optimal $\hat{b}$ and the corresponding cost for a given bucket takes time $O(n_b \log n_b |\mathcal{V}|)$. We can then apply the dynamic programming solution, since the principle of optimality holds over this error objective. Because of the structure of the cost function, it suffices to move from the tuple pdf model to the induced value pdf, and so we conclude,

Theorem 6: The optimal B bucket histogram under maximum-absolute-error (MAE) and maximum-absolute-relative-error (MARE) over data in either the tuple or value pdf models can be found in time $O(n^2(B + n \log n |\mathcal{V}|))$.

[1]The alternate formulation, where we seek to minimize the expectation of the maximum error, is also plausible, and worthy of further study.

[2]The same underlying optimization problem arises in a weighted histogram context—see [28] for the full details.

IV. Haar Wavelets on Probabilistic Data

A. SSE-*Optimal Wavelet Synopses.*

Any input defining a distribution over original g_i values also implies a distribution over Haar wavelet coefficients c_i. In particular, we have a possible-worlds distribution over Haar DWT coefficients, with $c_i(W)$ denoting the instantiation of c_i in world W (defined by the $g_i(W)$s). Our goal is to pick B coefficient indices $\mathcal{I}$ and corresponding coefficient values $\hat{c}_i$ for each $i \in \mathcal{I}$ to minimize the expected SSE in the data approximation. By Parseval's theorem [25] and linearity of the Haar transform in each possible world, the SSE of the data approximation is the SSE in the approximation of the normalized wavelet coefficients. By linearity of expectation, the expected SSE for the resulting synopsis $\mathcal{S}_w(\mathcal{I})$ is:

$$\mathsf{E}_{\mathcal{W}}[\mathsf{SSE}(\mathcal{S}_w(\mathcal{I}))] = \sum_{i \in \mathcal{I}} \mathsf{E}_{\mathcal{W}}[(c_i - \hat{c}_i)^2] + \sum_{i \notin \mathcal{I}} \mathsf{E}[(c_i)^2].$$

Suppose we are to include i in our index set $\mathcal{I}$ of selected coefficients. Then, the optimal setting of $\hat{c}_i$ is the expected value of the i^{th} (normalized) Haar wavelet coefficient, by the same argument style as Fact 1. That is,

$$\mu_{c_i} = \mathsf{E}_{\mathcal{W}}[c_i] = \textstyle\sum_{w_j} \Pr[c_i = w_j] \cdot w_j,$$

computed over the set of values taken on by the coefficients, w_j. Further, by linearity of expectation and the fact that the Haar wavelet transform can be thought of as a linear operator H applied to the input vector A, we have

$$\mu_{c_i} = \mathsf{E}_{\mathcal{W}}[H_i(A)] = H_i(\mathsf{E}_{\mathcal{W}}[A]).$$

In other words, we can find the μ_{c_i}'s by computing the wavelet transform of the expected frequencies, $\mathsf{E}_{\mathcal{W}}(g_i)$. So, the μ_{c_i}'s can be computed with linear effort from the input, in either tuple pdf or value pdf form. Based on the above observation, we can rewrite the expected SSE as:

$$\mathsf{E}_{\mathcal{W}}[\mathsf{SSE}(\mathcal{S}_w(\mathcal{I}))] = \textstyle\sum_{i \in \mathcal{I}} \sigma_{c_i}^2 + \sum_{i \notin \mathcal{I}} \mathsf{E}[(c_i)^2],$$

where $\sigma_{c_i}^2 = \mathsf{Var}_{\mathcal{W}}[c_i]$ is the variance of c_i. From the above expression, it is clear that the optimal strategy is to pick the B coefficients giving the largest reduction in the expected SSE (since there are no interactions across coefficients); furthermore, the "benefit" of selecting coefficient i is exactly $\mathsf{E}[(c_i)^2] - \sigma_{c_i}^2 = \mu_{c_i}^2$. Thus, the thresholding scheme that optimizes expected SSE is to simply select the B Haar coefficients with the largest (absolute) expected normalized value. (This scheme naturally generalizes the conventional deterministic SSE thresholding case (Section II-B).)

Theorem 7: With $O(n)$ time and space, we can compute the optimal SSE wavelet representation of probabilistic data in the tuple and value pdf models.

B. Wavelet Synopses for non-SSE Error.

The DP recurrence formulated over the Haar coefficient error tree for non-SSE error metrics in the deterministic case (Section II-B) extends naturally to the case of probabilistic data. The only change is that we now define $\text{OPTW}[j, b, v]$ to denote the *expected* optimal value for the error metric of interest under the same conditions as the deterministic case. The recursive computation steps remain exactly the same. The interesting point with the coefficient-tree DP recurrence is that almost all of the actual error computation takes place at the *leaf* (i.e., data) nodes of the tree — the DP recurrences combine these computed error values appropriately in a bottom-up fashion. For deterministic data, the error at a leaf node i with an incoming value of v from its parents is just the point error metric of interest with $\hat{g}_i = v$; that is, for leaf i, we compute $\text{OPTW}[i, 0, v] = \texttt{err}(g_i, v)$ which can be done in $O(1)$ time (note that leaf entries in $\text{OPTW}[]$ are only defined for $b = 0$ since space is never allocated to leaves).

In the case of probabilistic data, such leaf-error computations are a little more complicated since we now need to compute the *expected point-error value*

$$\mathsf{E}_{\mathcal{W}}[\texttt{err}(g_i, v)] = \sum_{W} \Pr[W] \cdot \texttt{err}(g_i(W), v)$$

over all possible worlds $W \in \mathcal{W}$. But this computation can still be done in $O(1)$ time assuming precomputed data structures similar to those we have derived for error objectives in the histogram case. To illustrate the main ideas, consider the case of *absolute relative error metrics*, i.e., $\texttt{err}(g_i, \hat{g}_i) = w(g_i) \cdot |g_i - \hat{g}_i|$ where $w(g_i) = 1/\max\{c, |g_i|\}$. Then, we can expand the expected error at g_i as follows:

$$\begin{aligned}\text{OPTW}[i, 0, v] =& \mathsf{E}_{\mathcal{W}}[w(g_i) \cdot |g_i - v|] \\ =& \sum_{v_j \in \mathcal{V}} \Pr[g_i = v_j] w(v_j) \cdot \begin{cases} (v - v_j) & \text{if } v > v_j \\ (v_j - v) & \text{if } v \leq v_j \end{cases}\end{aligned}$$

where, as earlier, $\mathcal{V}$ denotes the set of possible values for any frequency random variable g_i. So we have an instance of a sum-absolute-relative-error problem, since the form of this optimization matches that in Section III-C. By precomputing appropriate arrays of size $O(|\mathcal{V}|)$ for each i, we can search for the optimal "split point" $v_{j'} \in \mathcal{V}$ in time $O(\log |\mathcal{V}|)$.

The above precomputation ideas naturally extend to other error metrics as well, and allow us to easily carry over the algorithms and results (modulo the small $O(\log |\mathcal{V}|)$ factor above) for the *restricted* case, where all coefficient values are fixed, e.g., to their expected values as required for expected SSE minimization. The following theorem summarizes:

Theorem 8: Optimal restricted wavelet synopses for non-SSE error metrics can be computed over data presented in the (induced) value pdf model in time $O(n(|\mathcal{V}| + n \log |\mathcal{V}|))$.

For the *unrestricted* case, additional work is needed in order to effectively bound and quantize the range of possible coefficient values to consider in the case of probabilistic data at the leaf nodes of the tree. One option is to consider pessimistic coefficient-range estimates (e.g., based on the minimum/maximum possible frequency values); another option would be to employ some tail bounds on the g_i's (e.g., Chernoff bounds since tuples can be seen as binomial variables) to derive *tighter, high-probability* ranges for coefficient values.

V. Experiments

We implemented our algorithms in C, and carried out a set of experiments to compare the quality and scalability of our results against those from naively applying methods designed for deterministic data. Experiments were performed on a desktop 2.4GHz machine with 2GB RAM.

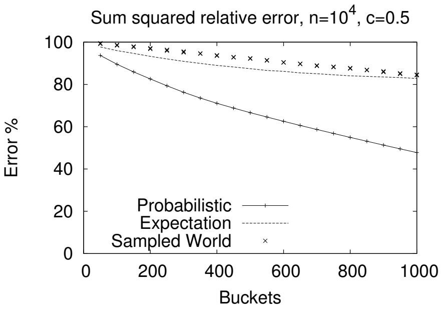
Data Sets. We experimented using a mixture of real and synthetic data sets. The real dataset came from the MystiQ project[3] which includes approximately $m = 127,000$ tuples describing $27,700$ distinct items. These correspond to links between a movie database and an e-commerce inventory, so the tuples for each item define the distribution of the number of expected matches. This uncertain data provides input in the basic model. Synthetic data was generated using the MayBMS [3] extension to the TPC-H generator [4]. We used the lineitem-partkey relation, where the multiple possibilities for each uncertain item are interpreted as tuples with uniform probability over the set of values in the tuple pdf model.

Sampled Worlds and Expectation. We compare our methods to the two naive methods using deterministic techniques discussed in Section II-C. The first is to simply sample a possible world, and compute the (optimal) synopsis for this deterministic sample, as in [12]. The second is to compute the expected frequency of each item, and build the synopsis of this deterministic input. This is equivalent to sampling many possible worlds, combining and scaling the frequencies of these, and building the summary of the result. For consistency, we use the same code to compute the respective synopses over both probabilistic and certain data, since deterministic data is a special case of probabilistic data in the value pdf model.

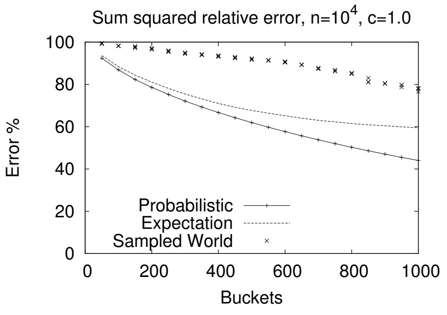
A. Histograms on Probabilistic Data

We use our methods described in Section III to build the histogram over n items using B buckets, and compute the cost of the histograms under the relevant metric (e.g. SSE, SARE, etc.). Observe that, unlike in the deterministic case, a histogram with $B = n$ buckets does not have zero error: we have to choose a fixed representative $\hat{b}$ for each bucket, so any bucket with some uncertainty will have a contribution to the expected cost. We therefore compute the percentage error of a given histogram as the fraction of the cost difference between the one bucket histogram (largest achievable error) and the n bucket histogram (smallest achievable error).

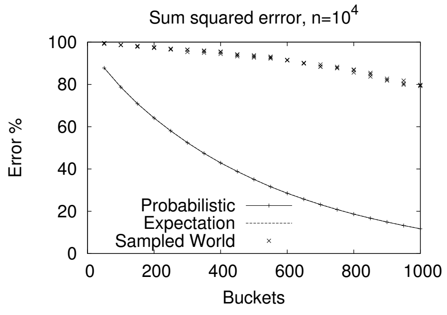
[3]`http://www.cs.washington.edu/homes/suciu/project-mystiq.html`
[4]`www.cs.cornell.edu/database/maybms/`



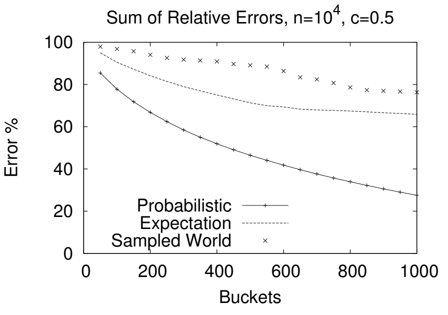
(a) Squared relative error with $c = 0.5$



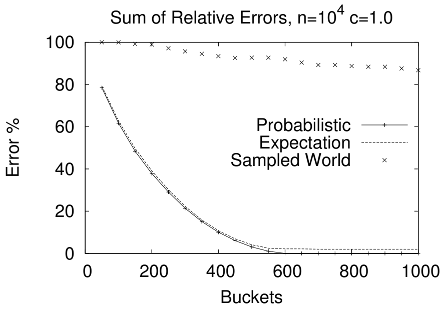
(b) Squared relative error with $c = 1.0$



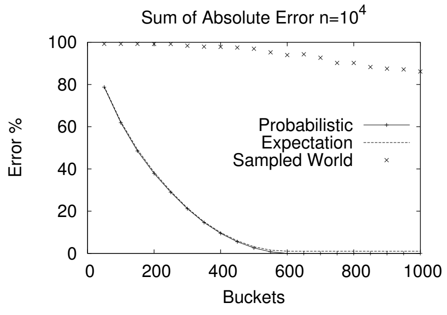
(c) Squared error



(d) Relative error with $c = 0.5$



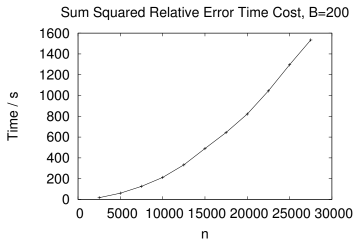
(e) Relative error with $c = 1.0$



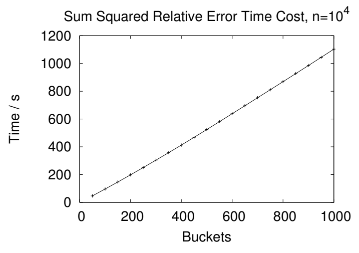
(f) Absolute error

Fig. 2. Results on Histogram Computation

Quality. For uniformity, we show results for the MystiQ movie data set; results on synthetic data were similar, and are omitted for space reasons. The quality of the different methods on the same $n = 10,000$ distinct data items is shown in Figure 2. In each case, we measured the cost for using up to 1000 buckets over the four cumulative error measures: SSE, SSRE, SAE, and SARE. We show two values of the sanity constant for relative error, $c = 0.5$ and $c = 1.0$. Since our results show that the dynamic programming finds the optimum set of buckets, there is no surprise that the cost is always smaller than the two naive methods. Figure 2(a) shows a typical case for relative error: the probabilistic method is appreciably better than using the expected costs, which in turn is somewhat better than building the histogram of a sampled world. We show the results for three independent samples to show that there is fairly little variation in the cost. For SSE and SAE (Figure 2(c) and 2(f)), while using a sampled world is still poor, the cost of using the expectation is very close to that of our probabilistic method. The reason is that the histogram obtains the most benefit by putting items with similar behavior in the same bucket, and on this data, the expectation is a good indicator of behavioral similarity. This is not always the case, and indeed, Figure 2(f) shows that while our method obtains the smallest possible error with about 600 buckets, using the expectation never finds this solution. Other values of c tend to vary smoothly between two extremes: increasing c allows the expectation method to get closer to the probabilistic solution (as in Figure 2(e)). This is because as c approaches the maximum achievable frequency of any item, there is no longer any dependency on the frequency, only on c, and so the cost function is essentially a scaled version of the squared error or absolute error respectively. Reducing c towards 0 further disadvantages the expectation method, and it has close to 100% error even when a very large number of buckets are provided; meanwhile, the probabilistic method smoothly reduces in error down to zero.



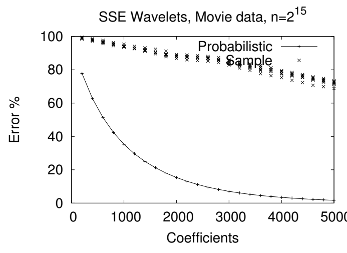
(a) Time as n varies



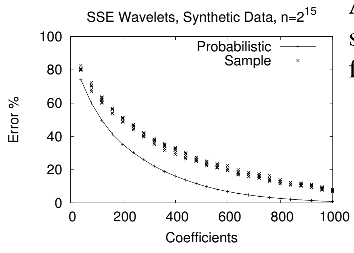
(b) Time as B varies

Fig. 3. Histogram Timing Costs

Scalability. Figure 3 shows the time cost of our methods. We show the results for SSE, although the results are very similar for other metrics, due to a shared code base. We see a strong linear dependency on the number of buckets, B, and a close to quadratic dependency on n (since as n doubles, the time cost slightly less than quadruples). This confirms our analysis that shows the cost is dominated by an $O(Bn^2)$ term. The time to apply the naive methods is almost identical, since they both ultimately rely on solving a dynamic program of the same size, which dwarfs any reduced linear preprocessing cost. Therefore, the cost is essentially the same as for deterministic



(a) Wavelets on real data



(b) Wavelets on synthetic data

Fig. 4. Results on Wavelet Computation

data. The time cost is acceptable, but it suggests that for larger relations it will be advantageous to pursue the faster approximate solutions outlined in Section III-E.

B. Wavelets on Probabilistic Data

We implemented our methods for computing wavelets under the SSE objective. Here, the analysis shows that the optimal solution is to compute the wavelet representation of the expected data (since this is equivalent to generating the expected values of the coefficients, due to linearity of the wavelet transform function), and then pick the B largest coefficients. We contrast to the effect of sampling possible worlds and picking the coefficients corresponding to the largest coefficients of the sampled data. We measure the error by computing the sum of the square of the μ_{c_i}'s not picked by the method, and expressing this as a percentage of the sum of all such μ_{c_i}'s, since our analysis demonstrates that this is the range of possible error (Section IV-A). Figures 4(a) and 4(b) shows the effect of varying the number of coefficients B on the real and synthetic data sets: while increasing the number of coefficients improves the cost in the sampled case, it is much more expensive than the optimal solution. Both approaches take the same amount of time, since they rely on computing a standard Haar wavelet transform of certain deterministic data: it takes linear time to produce the expected values and then compute the coefficients; these are sorted to find the B largest. This took much less than a second on our experimental set up.

VI. Concluding Remarks

We have introduced the probabilistic data reduction problem, and given results for a variety of cumulative and maximum error objectives, for both histograms and wavelets. Empirically, we see that the optimal synopses can accurately summarize the data and are significantly better than simple heuristics. It remains to better understand how to approximately represent and process probabilistic data, and to extend our results to even more expressive models. So far, we have focused on the foundational one-dimensional problem, but is also important to study multi-dimensional generalizations. The error objective formulations we have analyzed implicitly assume uniform workloads for point queries, and so it remains to address the case when in addition to a distribution over the input data, there is also a distribution over the queries.

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